Exercise 68

(a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200+t}$$

(b) What happens to the concentration as $t \to \infty$?

Solution

The concentration of salt at time t is the ratio of mass m to volume V at time t.

$$C(t) = \frac{m(t)}{V(t)}$$

The volume is 5000 L initially and increases by 25 L per minute.

$$V(t) = 5000 + 25t$$

Apply the law of conservation of mass to determine $m(t)$.

$$\begin{aligned}\begin{bmatrix}\text{rate that salt}\\ \text{accumulates in tank}\end{bmatrix} &= \big[\text{rate of salt in}\big] - \big[\text{rate of salt out}\big]\\ &= \big[\text{concentration in}\big] \times \big[\text{volumetric flow in}\big] - 0\\ &= \left(30\ \frac{\text{g}}{\text{L}}\right)\left(25\ \frac{\text{L}}{\text{min}}\right)\\ &= 750\ \frac{\text{g}}{\text{min}}\end{aligned}$$

The mass of salt initially is zero because the water is pure, so

$$m(t) = 0 + 750t = 750t.$$

Consequently, the concentration at time t in grams per liter is

$$\begin{aligned}C(t) = \frac{m(t)}{V(t)} &= \frac{750t}{5000+25t}\\ &= \frac{750t}{5000+25t} \cdot \frac{\frac{1}{25}}{\frac{1}{25}}\\ &= \frac{30t}{200+t}.\end{aligned}$$

In the limit as $t \to \infty$, the concentration becomes

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} \frac{30t}{200+t} = \lim_{t\to\infty} \frac{30}{\frac{200}{t}+1} = \frac{30}{0+1} = 30\ \frac{\text{g}}{\text{L}}.$$